

The Gauss-Bonnet-Grotemeyer Theorem in spaces of constant curvature *

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Abstract

In 1963, K.P. Grotemeyer proved an interesting variant of the Gauss-Bonnet Theorem. Let M be an oriented closed surface in the Euclidean space \mathbb{R}^3 with Euler characteristic $\chi(M)$, Gauss curvature G and unit normal vector field \vec{n} . Grotemeyer's identity replaces the Gauss-Bonnet integrand G by the normal moment $(\vec{a} \cdot \vec{n})^2 G$, where a is a fixed unit vector: $\int_M (\vec{a} \cdot \vec{n})^2 G dv = \frac{2\pi}{3} \chi(M)$. We generalize Grotemeyer's result to oriented closed even-dimensional hypersurfaces of dimension n in an $(n+1)$ -dimensional space form $N^{n+1}(k)$.

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1. Introduction

In 1963, K.P. Grotemeyer proved the following interesting result:

Theorem 1 [Gr]) *Let M be an oriented closed surface in 3-dimensional Euclidean space \mathbb{R}^3 with Gauss curvature G and a unit normal vector field \vec{n} . Then for any fixed unit vector \vec{a} in \mathbb{R}^3 , we have*

$$\int_M (\vec{a} \cdot \vec{n})^2 G dv = \frac{2\pi}{3} \chi(M), \quad (1.1)$$

where $\vec{a} \cdot \vec{n}$ denotes the inner product of \vec{a} and \vec{n} , $\chi(M)$ is the Euler characteristic of M .

Remark 1.1 Let $\{E_1, E_2, E_3\}$ be a fixed orthogonal frame in \mathbb{R}^3 and choose $\vec{a} = E_i$. We have

$$\int_M (E_i \cdot \vec{n})^2 G dv = \frac{2\pi}{3} \chi(M), \quad i = 1, 2, 3 \quad (1.2)$$

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Noting that $\sum_i (E_i \cdot \vec{n})(E_i \cdot \vec{n}) = \vec{n} \cdot \vec{n} = 1$, we obtain the following Gauss-Bonnet formula via summation of (1.2) over i from 1 to 3:

Corollary 1(Gauss-Bonnet Theorem). *Under the same hypothesis of Theorem 1, we have*

$$\int_M G dv = 2\pi\chi(M). \quad (1.3)$$

Thus we can consider Grotemeyer's Theorem 1 as an extended form of the Gauss-Bonnet Theorem.

Let n be even and let $N^{n+1}(k)$ be an $(n+1)$ -dimensional simply connected Riemannian manifold of constant sectional curvature k . That is, $N^{n+1}(k) = \mathbb{R}^{n+1}$ if $k = 0$; $N^{n+1}(k) = S^{n+1}(\frac{1}{\sqrt{k}})$, an $(n+1)$ -dimensional sphere space with radius $\frac{1}{\sqrt{k}}$ if $k > 0$; $N^{n+1}(k) = H^{n+1}(-\frac{1}{\sqrt{-k}})$, an $(n+1)$ -dimensional hyperbolic space with, as Bolyai would say, radius $\sqrt{-1}/\sqrt{k}$ if $k < 0$. We will often call $N^{n+1}(k)$ a *space form*. We will view $N^{n+1}(k)$ as standardly imbedded in an appropriate linear space $L_{n+1}(k)$ (\mathbb{R}^{n+2} if $k > 0$, $\mathbb{R}^{n+1,1}$ if $k < 0$ and \mathbb{R}^{n+1} if $k = 0$).

This will enable us to define functions on M such as $(\vec{a} \cdot \vec{n})$, where \vec{a} is a fixed vector in the ambient linear space, \vec{n} is a normal vector field on M , and (\cdot) denotes the inner product on the ambient linear space. The generalized Grotemeyer Theorem we have in mind can be stated as follows:

Theorem 2 *Let n even, $n \geq 2$. Let $\vec{x} : M \rightarrow N^{n+1}(k)$ be an immersed n -dimensional oriented closed hypersurface in the $(n+1)$ -dimensional space form $N^{n+1}(k)$, with Euler characteristic $\chi(M)$, Gauss-Kronecker curvature G and unit normal vector field \vec{n} . Assume that $N^{n+1}(k)$ is standardly imbedded in the linear space $L_{n+1}(k)$. Then for any fixed unit vector \vec{a} in $L_{n+1}(k)$ we have*

$$\begin{aligned} \int_M (\vec{a} \cdot \vec{n})^2 G dv &= \frac{1}{n+1} \left[\frac{\text{vol} S^n(1)}{2} \chi(M) - \sum_i c_i k^i \int_M K_{n-2i} dv \right] \\ &+ \frac{k}{n+1} \int_M (\vec{a} \cdot \vec{n})(\vec{a} \cdot \vec{x}) K_{n-1} dv - \frac{k}{n+1} \int_M (\vec{a} \cdot \vec{x})^2 G dv, \end{aligned} \quad (1.4)$$

where the c_i are constants that depend only on the dimension n and K_i is the i -th mean curvature of M .

In the case $n = 2$ in the Theorem above, we obtain

Corollary 2 *Let M be an oriented closed surface in the 3-dimensional space form $N^3(k)$ with extrinsic curvature G and unit normal vector field \vec{n} . Then for any fixed unit vector \vec{a} in the linear space $L_3(k)$ we have*

$$\begin{aligned} \int_M (\vec{a} \cdot \vec{n})^2 G dv &= \frac{2\pi}{3} \chi(M) - \frac{k}{3} \text{vol}(M) \\ &+ \frac{k}{3} \int_M (\vec{a} \cdot \vec{n})(\vec{a} \cdot \vec{x}) K_1 dv - \frac{k}{3} \int_M (\vec{a} \cdot \vec{x})^2 G dv, \end{aligned} \quad (1.5)$$

where K_1 is the mean curvature of M and $\chi(M)$ is the Euler characteristic of M .

Remark 1.2 Our Corollary reduces to Grotemeyer's original theorem in the case $k = 0$.

Remark 1.3 In the case $k = 0$ and $n \geq 3$, Theorem 2 was proved by B. -Y. Chen in [Ch] by a different method.

Remark 1.4. We can recover the standard Gauss-Bonnet Theorem from our Theorem as follows. Let m be the dimension of the linear space $L_{n+1}(k)$. (Thus $m = n + 1$ in the flat case, $m = n + 2$ in the positive and negatively curved cases.) Let $\{E_1, \dots, E_m\}$ be a fixed orthonormal frame in $L_{n+1}(k)$; choose $\vec{a} = E_i$. Then

$$\begin{aligned} \int_M (E_i \cdot \vec{n})^2 G dv &= \frac{1}{n+1} \left[\frac{\text{vol} S^n(1)}{2} \chi(M) - \sum_i c_i k^i \int_M K_{n-2i} dv \right] \\ &+ \frac{k}{n+1} \int_M (E_i \cdot \vec{n})(E_i \cdot x) K_{n-1} dv \\ &- \frac{k}{n+1} \int_M G(E_i \cdot x)^2 dv, \quad (i = 1, 2, \dots) \end{aligned} \quad (1.6)$$

Noting $\sum_i (E_i \cdot \vec{n})(E_i \cdot \vec{n}) = \vec{n} \cdot \vec{n} = 1$ and $\sum_i (E_i \cdot \vec{n})(E_i \cdot x) = \vec{n} \cdot x = 0$, we obtain the following Gauss-Bonnet formula by summing of (1.6) over all appropriate i :

Corollary 3 (Gauss-Bonnet Theorem). *Under the same hypothesis of Theorem 2, we have*

$$\int_M G dv = \frac{\text{vol} S^n(1)}{2} \chi(M) - \sum_i c_i k^i \int_M K_{n-2i} dv \quad (1.7)$$

where $\chi(M)$ is the Euler characteristic of M , constants c_i depends only on dimension n , and K_i is the i -th mean curvature of M .

So we can view Theorem 2 as an extended form of the Gauss-Bonnet Theorem.

2. Reilly's operator and its properties

In order to prove Theorem 2, we need to recall Reilly's operator and its properties.

Let (M, g) be a closed n -dimensional Riemannian manifold, let $\{e_1, \dots, e_n\}$ be a local orthonormal frame field in M with dual coframe field $\{\theta_1, \dots, \theta_n\}$. Given a symmetric tensor $\phi = \sum_{i,j} \phi_{ij} \theta_i \theta_j$ defined on M we define a second order differential operator

$$\square \equiv \square_\phi : C^\infty(M) \rightarrow C^\infty(M), \quad \square f = \sum_{i,j} \phi_{ij} f_{ij} \quad (2.1)$$

where f_{ij} are the components of the second covariant differential of f , as follows:

$$df = \sum_i f_i \theta_i, \quad df_i + \sum_j f_j \theta_{ji} = \sum_j f_{ij} \theta_j, \quad (2.2)$$

where $\{\theta_{ij}\}$ is the Levi-Civita connection of g .

For the following criterion for self adjointness of the of the operator \square see Cheng-Yau [CY] or Li [L1],[L2].

Proposition 2.1 *Let M be a closed orientable Riemannian manifold with symmetric tensor $\phi = \sum_{i,j} \phi_{ij} \theta_i \theta_j$. Then \square is a selfadjoint operator if and only if*

$$\sum_{j=1}^n \phi_{ij,j} = 0, \quad 1 \leq i \leq n. \quad (2.3)$$

Here $\phi_{ij,k}$ is the derivative of the tensor ϕ_{ij} in the direction e_k .

Remark 2.1 We call \square *the Cheng-Yau operator*. It was introduced by S.Y. Cheng and S.T. Yau in 1977 [CY]. If $\phi = \sum_{i,j} \phi_{ij} \theta_i \theta_j$ satisfies the Cheng-Yau condition (2.3), then

$$\square f = \sum_{i,j} \phi_{ij} f_{ij} = \sum_{i,j} (\phi_{ij} f_i)_j = \operatorname{div}(\phi \nabla f).$$

Let $x : M \rightarrow N^{n+1}(k)$ be an n -dimensional closed hypersurface in an $(n+1)$ -dimensional space form of constant sectional curvature k . Let (h_{ij}) be the components of the second fundamental form of M . We recall the *Reilly operator*, which is a second order differential operator $L_r : C^\infty(M) \rightarrow C^\infty(M)$ defined by

$$L_r f = \sum_{i,j} T_{ij}^r f_{ij}, \quad f \in C^\infty(M), \quad (2.4)$$

where T_{ij}^r is given by

$$T_{ij}^0 = \delta_{ij}, \quad T_{ij}^r = K_r \delta_{ij} - \sum_k h_{ik} T_{kj}^{r-1}, \quad r = 1, 2, \dots, n. \quad (2.5)$$

(See Reilly [Re], Rosenberg [Ro] or Barbosa-Colares [BC].)

Denote the r^{th} mean curvature of M by

$$K_r = \sum_{i_1 < \dots < i_r} k_{i_1} \dots k_{i_r}, \quad B = (h_{ij}) = (k_i \delta_{ij}). \quad (2.6)$$

We note that the Gauss-Kronecker curvature of M is $G \equiv K_n$.

Definition 2.1 ([Re]) The r -th Newton transformation, $r \in \{0, 1, \dots, n\}$ is the linear transformation

$$T_r = K_r I - K_{r-1} B + \dots + (-1)^r B^r, \quad (2.7)$$

i.e.,

$$T_{ij}^r = K_r \delta_{ij} - K_{r-1} h_{ij} + \dots + (-1)^r \sum_{j_1, \dots, j_r} h_{ij_1} h_{j_1 j_2} \dots h_{j_r j}. \quad (2.7)'$$

If $I \equiv i_1, \dots, i_q$ and $J \equiv j_1, \dots, j_q$ are multi-indices of integers between 1 and n , define

$$\delta_I^J = \begin{cases} 1, & \text{if } i_1, \dots, i_q \text{ are distinct and } J \text{ is an even permutation of } I \\ -1, & \text{if } i_1, \dots, i_q \text{ are distinct and } J \text{ is an odd permutation of } I \\ 0, & \text{otherwise.} \end{cases}$$

Then we have (see Reilly [Re])

$$K_r = \frac{1}{r!} \sum \delta_{i_1 \dots i_r}^{j_1 \dots j_r} h_{i_1 j_1} \dots h_{i_r j_r}. \quad (2.8)$$

Proposition 2.2 *The matrix of T_r is given by*

$$T_{ij}^r = \frac{1}{r!} \sum \delta_{i_1 \dots i_r i}^{j_1 \dots j_r j} h_{i_1 j_1} \dots h_{i_r j_r}. \quad (2.9)$$

Proposition 2.3 *For each r , we have*

- (1) $\operatorname{div} T_r = \sum_j T_{ij,j}^r = 0$,
- (2) Newton's formula: $\operatorname{trace}(BT_r) = (r+1)K_{r+1}$,
- (3) $\operatorname{trace}(T_r) = (n-r)K_r$

Proposition 2.4 *Let $\vec{x} : M \rightarrow N^{n+1}(k)$ be an n -dimensional hypersurface with unit normal vector field \vec{n} . Then we have*

$$x_i = e_i, \quad \vec{n}_i = - \sum_j h_{ij} e_j, \quad x_{ij} = h_{ij} \vec{n} - kx \delta_{ij}. \quad (2.10)$$

$$L_r x = (r+1)K_{r+1} \vec{n} - (n-r)kK_r x, \quad (2.11)$$

Proof. Let \vec{a} be a fixed vector in $L_n(k)$. Write

$$f = \vec{n} \cdot \vec{a}, \quad g = \vec{x} \cdot \vec{a}. \quad (2.12)$$

Then (2.11) is equivalent to

$$L_r g = (r+1)K_{r+1} f - (n-r)kK_r g. \quad (2.11)'$$

Choosing an orthonormal frame $\{e_1, \dots, e_n, \vec{n}\}$ and their dual frame $\{\theta_1, \dots, \theta_n, \theta_{n+1}\}$ along M in $N^{n+1}(k)$, we have the structure equations

$$dx = \sum_i \theta_i e_i, \quad de_i = \sum_j \theta_{ij} e_j + \sum_j h_{ij} \theta_j \vec{n} - kx \theta_i, \quad d\vec{n} = - \sum_{i,j} h_{ij} \theta_j e_i. \quad (2.13)$$

Here we have sometimes abbreviated \vec{x} as merely x , for simplicity. By use of (2.13) and through a direct calculation we get

$$g_i = e_i \cdot \vec{a}, \quad g_{ij} = fh_{ij} - kg\delta_{ij}. \quad (2.14)$$

By use of proposition 2.3 and (2.14), we get

$$L_r g = \sum_{i,j} T_{ij}^r g_{ij} = \sum_{ij} T_{ij}^r h_{ij} f - kg \sum_{i,j} T_{ij}^r \delta_{ij} = (r+1)K_{r+1}f - k(n-r)gK_r.$$

Thus we have proved (2.11)', which is equivalent to (2.11). Similarly, from definitions of f_i , we get by use of (2.13)

$$f_i = - \sum_j h_{ij} (e_j \cdot \vec{a}). \quad (2.15)$$

Because \vec{a} is arbitrary, we have proved (2.10) from (2.14) and (2.15).

Proposition 2.5 *Let M be an n -dimensional oriented closed hypersurface in $(n+1)$ -dimensional space form $N^{n+1}(k)$. Then for any smooth functions f and g on M we have*

$$\int_M g L_{n-1} f dv = \int_M f L_{n-1} g dv, \quad \int_M L_{n-1} f dv = 0. \quad (2.16)$$

Proof. Choosing $r = n-1$ in (1) of proposition 2.3, and using the criterion from proposition 2.1, we know that the operator L_{n-1} is a selfadjoint operator. Thus we obtain (2.16).

Proposition 2.6 *Let M be an n -dimensional hypersurface in $(n+1)$ -dimensional space form $N^{n+1}(k)$. Then we have*

$$G\delta_{ij} - \sum_k h_{ik} T_{kj}^{n-1} = 0. \quad (2.17)$$

Proof. Choosing $r = n-1$ in (2.5) and noting that $G = K_n$, we have

$$T_{ij}^n = G\delta_{ij} - \sum_k h_{ik} T_{kj}^{n-1}. \quad (2.18)$$

From the definition of T_{ij}^n in (2.9) and the definition of $\delta_{i_1 \dots i_n i}^{j_1 \dots j_n j}$, we have

$$T_{ij}^n = 0. \quad (2.19)$$

Now (2.17) follows from (2.18) and (2.19).

3. Proof of Theorem 2

Proposition 3.1 *Let $x : M \rightarrow N^{n+1}(k)$ be an n -dimensional oriented closed hypersurface in $(n+1)$ -dimensional space form $N^{n+1}(k)$. Assume M has Gauss-Kronecker curvature $G = K_n$ and a unit normal vector \vec{n} . Then for any fixed unit vector \vec{a} in $L_{n+1}(k)$, we have*

$$0 = (n+m) \int_M (\vec{a} \cdot \vec{n})^{m+1} G dv - m \int_M (\vec{a} \cdot \vec{n})^{m-1} G dv - k \int_M (\vec{a} \cdot \vec{n})^m (\vec{a} \cdot \vec{x}) K_{n-1} dv + mk \int_M (\vec{a} \cdot \vec{n})^{m-1} (\vec{a} \cdot \vec{x})^2 G dv, \quad (3.1)$$

where K_{n-1} is the $(n-1)$ -th mean curvature of M .

Proof. Write

$$f = q^m x, \quad q = \vec{a} \cdot \vec{n}. \quad (3.2)$$

By definition of the first derivative and the second derivative of f (see (2.2)), we have

$$f_i = (q^m)_i x + q^m x_i, \quad (3.3)$$

$$f_{ij} = (q^m)_{ij} x + (q^m)_i x_j + (q^m)_j x_i + q^m x_{ij}. \quad (3.4)$$

By definition of operator L_{n-1} , we have

$$L_{n-1}(f) = x L_{n-1}(q^m) + 2 \sum_{i,j} T_{ij}^{n-1} (q^m)_i x_j + q^m L_{n-1} x. \quad (3.5)$$

Let $r = n-1$ in (2.11). We have

$$L_{n-1} x = n G \vec{n} - k K_{n-1} x. \quad (3.6)$$

By Proposition 2.5, (3.6), (2.10) and proposition 2.6, we get by integrating (3.5) over M

$$\begin{aligned} 0 &= 2 \int_M q^m (L_{n-1} x) dv + 2 \int_M \sum_{i,j} T_{ij}^{n-1} (q^m)_i x_j dv \\ &= 2 \int_M q^m (n G \vec{n} - k K_{n-1} x) dv + 2 \int_M \sum_{i,j,k} T_{ij}^{n-1} m q^{m-1} [-h_{ik} (\vec{a} \cdot e_k)] e_j dv \\ &= 2 \int_M q^m (n G \vec{n} - k K_{n-1} x) dv - 2m \int_M q^{m-1} G \sum_j (\vec{a} \cdot e_j) e_j dv \\ &= 2 \int_M q^m (n G \vec{n} - k K_{n-1} x) dv - 2m \int_M q^{m-1} G [\vec{a} - (\vec{a} \cdot \vec{n}) \vec{n} - k (\vec{a} \cdot \vec{x}) \vec{x}] dv, \end{aligned} \quad (3.7)$$

that is, we obtain for $m = 1, 2, 3, \dots$

$$0 = (n+m) \int_M q^m G \vec{n} dv - m \int_M q^{m-1} G \vec{a} dv - k \int_M q^m K_{n-1} x dv + mk \int_M q^{m-1} (x \cdot \vec{a}) G x dv. \quad (3.8)$$

Taking the scalar product of \vec{a} with both sides of (3.8), we get Proposition 3.1.

Remark 3.1 Equation (3.1) was proved by Bang-Yen Chen in the case $k = 0$ by a different method.

Proof of Theorem 2 Choosing $m = 1$ in Proposition 3.1, we have

$$(n+1) \int_M (\vec{a} \cdot \vec{n})^2 G dv = \int_M G dv + k \int_M (\vec{a} \cdot \vec{n})(\vec{a} \cdot \vec{x}) K_{n-1} dv - k \int_M G (\vec{a} \cdot \vec{x})^2 dv. \quad (3.9)$$

Because M is a closed hypersurface in $N^{n+1}(k)$, the Gauss-Bonnet Theorem states in this case that

$$\int_M G dv = \frac{\text{vol} S^n(1)}{2} \chi(M) - \sum_i c_i k^i \int_M K_{n-2i} dv, \quad (3.10)$$

where $\chi(M)$ is the Euler characteristic of M , the constants c_i depend only on dimension n , and K_i is the i -th mean curvature of M . (See p. 1105 of [So]; c.f. [C1], [C2].) Inserting (3.10) into (3.9), we have proved our Theorem 2. On the other hand, by choosing $m = 0$ in (3.1), we have

Corollary 3.1 (Bivens [Bi]) *Let $x : M \rightarrow N^{n+1}(k)$ be an n -dimensional closed oriented hypersurface in $N^{n+1}(k)$. Then*

$$\int_M [n(\vec{a} \cdot \vec{n})G - k(\vec{a} \cdot \vec{x})K_{n-1}] dv = 0, \quad (3.11)$$

where \vec{a} is any fixed unit vector in the linear space $L_{n+1}(k)$, G is the Gauss-Kronecker curvature of M and K_{n-1} is the $(n-1)$ -th mean curvature of M .

Remark 3.2 Write $q = \vec{a} \cdot \vec{n}$, from Proposition 3.1, we have

$$\int_M q^m G dv = \frac{m-1}{n+m-1} \left[\int_M q^{m-2} G dv - k \int_M q^{m-2} (x \cdot \vec{a})^2 G dv + \frac{k}{m-1} \int_M q^{m-1} (x \cdot \vec{a}) K_{n-1} dv \right]. \quad (3.12)$$

By a direct calculation using (3.12), (3.10) and Corollary 3.1, we obtain

Proposition 3.2 *Let n and m be even. Under the same hypothesis of Proposition 3.1, we have*

$$\begin{aligned} & \int_M q^m G dv \\ = & \frac{(m-1)(m-3)\cdots 1}{(n+m-1)(n+m-3)\cdots(n+1)} \left[\frac{\text{vol} S^n(1)}{2} \chi(M) - \sum_i c_i k^i \int_M K_{n-2i} dv \right] \\ & - k \int_M \left[\frac{m-1}{n+m-1} q^{m-2} + \frac{(m-1)(m-3)}{(n+m-1)(n+m-3)} q^{m-4} + \cdots + \frac{(m-1)(m-3)\cdots 1}{(n+m-1)(n+m-3)\cdots(n+1)} \right] (\vec{x} \cdot \vec{a})^2 G dv \\ & + k \int_M \left[\frac{1}{n+m-1} q^{m-1} + \frac{m-1}{(n+m-1)(n+m-3)} q^{m-3} + \cdots + \frac{(m-1)(m-3)\cdots 3}{(n+m-1)(n+m-3)\cdots(n+1)} q \right] (\vec{x} \cdot \vec{a}) K_{n-1} dv. \end{aligned} \quad (3.13)$$

Also, for n even and m odd, we have

$$\begin{aligned} & \int_M q^m G dv \\ = & -k \int_M \left[\frac{m-1}{n+m-1} q^{m-2} + \frac{(m-1)(m-3)}{(n+m-1)(n+m-3)} q^{m-4} + \cdots + \frac{(m-1)(m-3)\cdots 2}{(n+m-1)(n+m-3)\cdots(n+2)} q \right] (\vec{x} \cdot \vec{a})^2 G dv \\ & + k \int_M \left[\frac{1}{n+m-1} q^{m-1} + \frac{m-1}{(n+m-1)(n+m-3)} q^{m-3} + \cdots + \frac{(m-1)(m-3)\cdots 2}{(n+m-1)(n+m-3)\cdots n} \right] (\vec{x} \cdot \vec{a}) K_{n-1} dv. \end{aligned} \quad (3.14)$$

Note: In the case $k = 0$, Proposition 3.2 was proved by Bang-Yen Chen; see Theorem 2 in [Ch].

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